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ELASTICA OF AN EULER ROD WITH CLAMPED ENDS

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The stability of the postcritical states of equilibrium of a flexible rod with clamped ends loaded by an axial force is analyzed. It is shown that the existing Lagrange elliptic-integral solution has bifurcation points and branches of solution that have not been investigated thus far.

Exact analytical solutions that govern the postcritical bending of rods (Euler elastica) were obtained by Lagrange in terms of elliptic integrals [1]. However, the stability of the postcritical states of equilibrium has not yet been investigated. Kuznetsov and Levyakov [2] analyzed the stability of a simply supported rod and found new bifurcation points and branches of solution. In the present paper, a similar analysis is performed for a rod with clamped ends.

We consider a straight elastic rod of a uniform rectangular cross section with clamped ends. We denote the rod length by l and the extensional and bending rigidities by EF and EI, respectively. The rod is loaded by the axial force P (Fig. 1). We investigate the plane elastica with the use of the numerical algorithm from [3], which is applicable to analysis of multiple branching solutions. The ratio of the height of the rod to its length was taken to be $2.5 \cdot 10^{-3}$, which made the axial line practically inextensible. As a numerical analysis has shown, the division of the rod into 50 equal finite elements provides high accuracy of the solution for a wide range of rod curvatures.

Figure 1 shows the nonlinear deformation characteristics of the rod ($P_{\rm cr} = 4\pi^2 EI/l^2$ is the first critical load, u is the displacement of the movable support, and w is the mid-span deflection of the rod). The solid curves refer to stable states and the dashed curves to unstable states. The points refer to the singular points of solution.

We analyze the results obtained. As the compressive load increases from zero to the value of $P = P_{\rm cr}$, the rod remains rectilinear and stable. The solution bifurcates at the bifurcation point B_1. With further increase in load, the bent configurations are stable (the curves B_1B_2 and B_1B_3 in Fig. 1). These findings are well known. However, as the calculations have shown, these curved configurations of the rod are stable only up to the point B_2 (B_3), where the secondary loss of stability occurs for $P = 2.1539P_{\rm cr}$. When this value of the load is reached, the rod configuration becomes 8-shaped (Fig. 2a). In this case, the second variation of the total potential energy of the rod is not positive definite, and, hence, this state of equilibrium is unstable.

According to the Lagrange solution, the force that brings the rod ends into coincidence is determined from the transcendental equation

$$2E(\pi/2, k) - F(\pi/2, k) = 0, \qquad P = (4/\pi^2)F^2(\pi/2, k)P_{\rm cr}, \tag{1}$$

in which $F(\pi/2, k)$ and $E(\pi/2, k)$ are the complete elliptic integrals of the first and second kinds, respectively, and k is the modulus of the elliptic integral. From (1), we find

$$k = 0.908908557, \qquad P = 2.18337905P_{\rm cr}. \tag{2}$$

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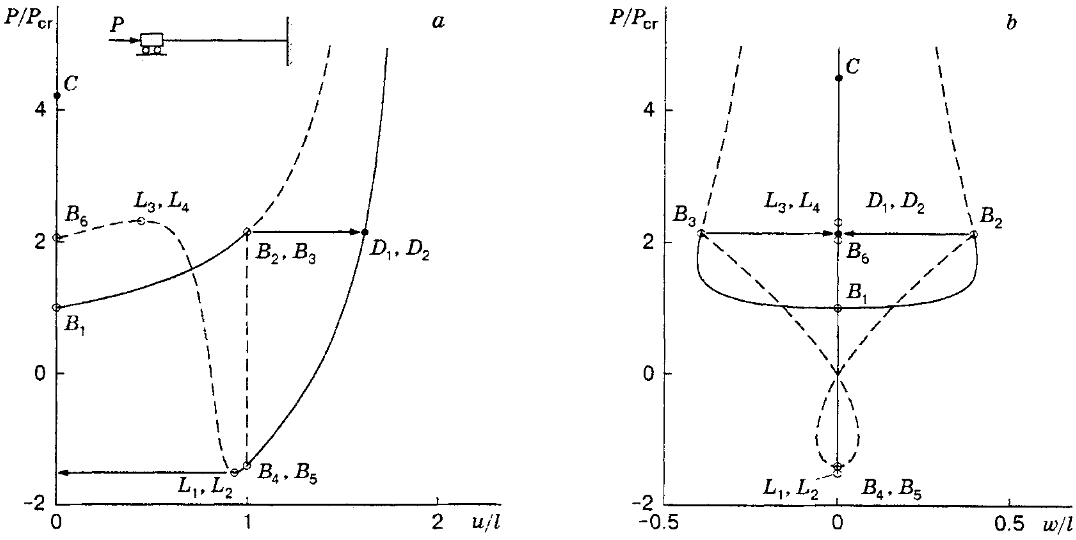

Fig. 1

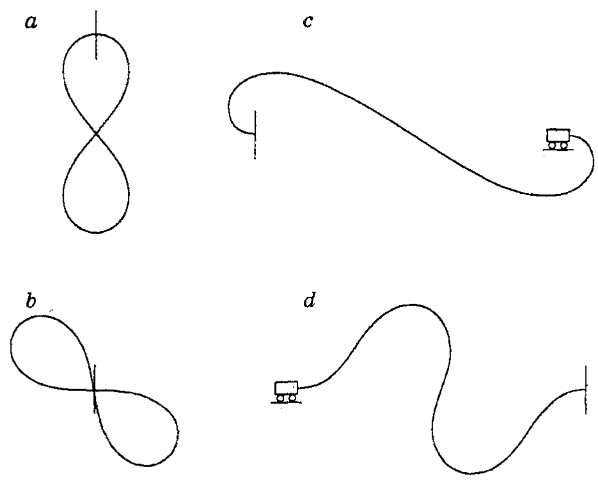

Fig. 2

The load of the secondary loss of stability, which was determined by the algorithm of [3], differs from the value in (2) by 1.3%. The branch of solution $B_2B_4B_3B_5B_2$ (see Fig. 1), which passes through the points B_2 and B_3, is a closed curve in the enhanced space of generalized coordinates and load parameter. This branch refers to the deformation upon which the 8-shaped configuration of the rod rotates. In the process, the strain state at each point of the rod varies cyclically, the potential energy of the rod being constant.

At the bifurcation points B_4 and B_5, the above-described branch intersects the new branch $D_1B_4L_1L_3B_6L_4L_2B_5D_2$, which includes the bifurcation points B_4, B_5, and B_6 and the limit points L_1, L_2, L_3, and L_4. The configuration of the deformed rod that corresponds to the point B_4 is shown in Fig. 2b. On this branch, the S-shaped configurations shown in Fig. 2c (the point D_1) are stable, and the configurations shown in Fig. 2d (the point L_3) are unstable. We note that this branch describes the rod deformation that corresponds to the development of the second buckling mode of the rectilinear rod for $P = 2.050\,12P_{\rm cr}$ (the bifurcation point B_6).

Thus, in contrast to the Lagrange solution B_1B_2 (or B_1B_3) and its smooth continuation, as a series of stable states of equilibrium at an increased load, the deformation develops according to the branch B_1B_2 (or B_1B_3) with a subsequent snap on the branch B_4D_1 (or B_5D_2).

TABLE 1

Bifurcation points	P/P_{cr}	u/l	w/l
B_1	1	0	0
B_2, B_3	2.1539	1	±0.3927
B_4, B_5	−1.4010	1	0
L_1, L_2	−1.5132	0.9347	0
L_3, L_4	2.3153	0.4451	0
B_6	2.0512	0	0

When the movable-end displacement and the mid-span deflection are used as the characteristic displacements, the projections of the singular points B_4 and B_5, L_1 and L_2, and L_3 and L_4 coincide (see Fig. 1). The configurations that correspond to the points L_2 and L_4 are the reflections of the configurations corresponding to the points L_1 and L_3, respectively. The configurations that refer to the bifurcation points B_5 and B_4 are the reflections in the vertical axis that passes through the immovable end of the rod.

Table 1 lists values of the dimensionless parameters that characterize rod deformations at the singular points of solution.

Thus, the stability analysis of the nonlinear solutions, including well-known solutions, has revealed new singular points and branches of solution that emanate from these points.

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